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Opportunities and Perils of Using Option Sensitivities

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ABSTRACT

Delta, gamma, theta, vega, and rho each express a dimension of risk (see Appendix A), and each dimension has its specific characteristics. If the option is deep-out-of-the-money, the delta will approach zero, and if the option is deep-in-the-money, it will approach one. Vega reaches its maximum value when the option has a long time to maturity and is approximately at-the-money. Gamma increases significantly immediately prior to maturity when the option is approximately at-the-money, and approaches zero when the option is going in- or out-of-the-money. Theta is usually large and negative when gamma is large and positive—and vice versa. These sensitivities play an important role in calculating the risk of positions in individual options. Clearly, when managing a large portfolio of options, an aggregate measure of risk is essential. For the majority of the sensitivities this can be accomplished by simple means. This article's main focus is on a new measure of volatility risk in option portfolios.

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I. RISK MANAGEMENT OF AN OPTION PORTFOLIO

When considering investment opportunities, or when searching for the best hedging strategy, market participants make active use of option sensitivities. The net risk exposure of a portfolio is found by separately adding each risk dimension for the individual options. Thus, delta for a portfolio of options is formally given as:

$$\Delta_{\text{Portfolio}} = \sum_{i=1}^{n} w_i \Delta_i$$

where w_i is the number of options in the portfolio with Δ_i. Total gamma, theta, and vega for the portfolio can be found in a similar manner. (As will be shown later, this does not necessarily pertain to vega for portfolios of options with different maturities.)

Table 1
Option Sensitivities of an Option Portfolio

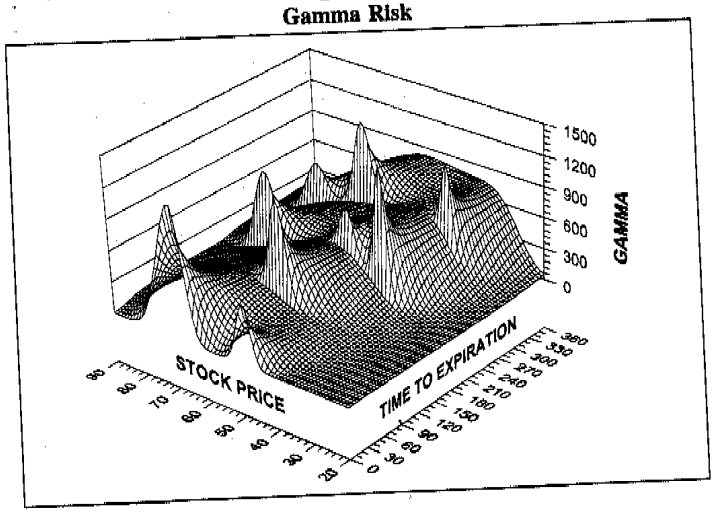
No. of Option Contracts	Days to Maturity	Strike	Call Value	Delta	Gamma	Theta/ 365	Vega/ 100
40	360	75	5.40	1,707.37	75.22	–67.59	934.85
80	360	70	6.91	4,044.90	153.01	–143.79	1,901.57
30	360	50	16.82	2,510.90	35.43	–48.85	440.31
50	270	55	12.02	3,767.11	87.33	–98.21	813.93
55	180	75	2.26	1,537.23	125.43	–97.64	779.41
20	180	50	13.62	1,713.27	30.68	–39.23	190.61
48	180	40	22.07	4,683.67	18.53	–59.16	115.17
30	90	75	0.72	437.03	65.78	–46.32	204.36
48	90	60	4.88	2,834.29	178.89	–148.25	555.79
25	90	35	25.85	2,498.91	0.37	–23.60	116.13
Stocks –25,735				–25,735.00			
Total Risk				–0.32	770.67	–772.64	?

Stock price 60, volatility 35%, risk–free interest rate 10%. One contract represents rights to 100 shares of the underlying stock.

When calculating the different risk parameters—delta, gamma, vega, theta, and rho—we should note that these express the risk–exposure at a given point in time. Maintaining an overview of the risk–exposures that may arise in different scenarios may cause great difficulty when managing portfolios in which the options have several different strikes and maturities. Without a sufficiently advanced risk–management tool, the effect of changes in the underlying pricing variables may come as a surprise even to experienced traders.

Table 1 shows an example of a stock options portfolio with seven different strikes and four different maturities. The different risk dimensions are calculated for each option by employing the Black and Scholes option pricing formula (1973). The portfolio is made delta–neutral by going 25,735 stocks short, resulting in a positive gamma exposure and a negative theta exposure. As mentioned above, the risk dimensions are dynamic variables that are influenced by all the pricing variables of the option. What will happen to our risk exposure if the stock price decreases 15 percent during a five–day period, or if the volatility increases by 4 percentage points and the stock price is up 10 percent? Several alternative scenarios arise. Three–dimensional graphics can serve as a useful tool in this kind

Figure 1a
Gamma Risk



of scenario–analysis. Figure 1a shows the gamma exposure of our portfolio when the stock price and time to maturity change, ceteris paribus.

The graph depicts the conditions under which our gamma exposure increases substantially. A graphic analysis like this enables us to detect opportunities and dangers far in advance. As mentioned earlier, gamma increases dramatically just before maturity when the option is at–the–money. Local peaks therefore occur for each strike, near to maturity. The delta and theta exposure of a portfolio of options can similarly be studied in Figures 1b and 1c.

II. ADDING VOLATILITY RISK—AN EXCEPTION

In Table 1, the delta, gamma, and theta exposure of the portfolio were found by adding the risk exposure of each option regardless of strike and time to maturity.

Figure 1b
Theta Risk

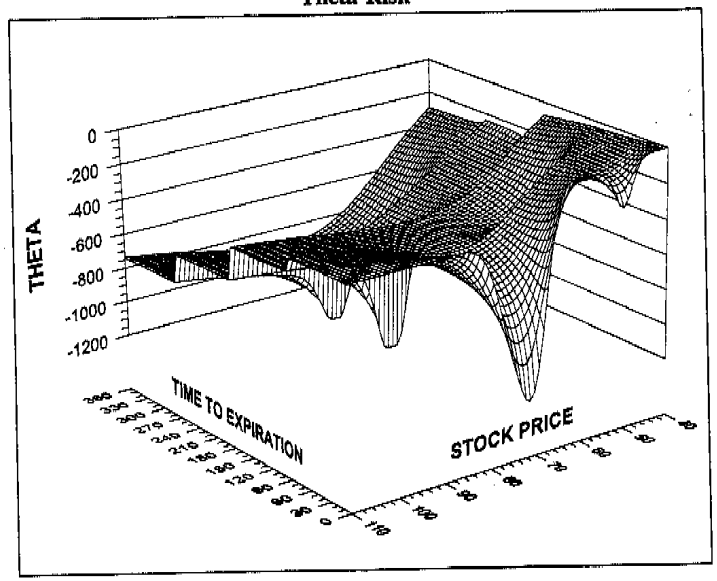
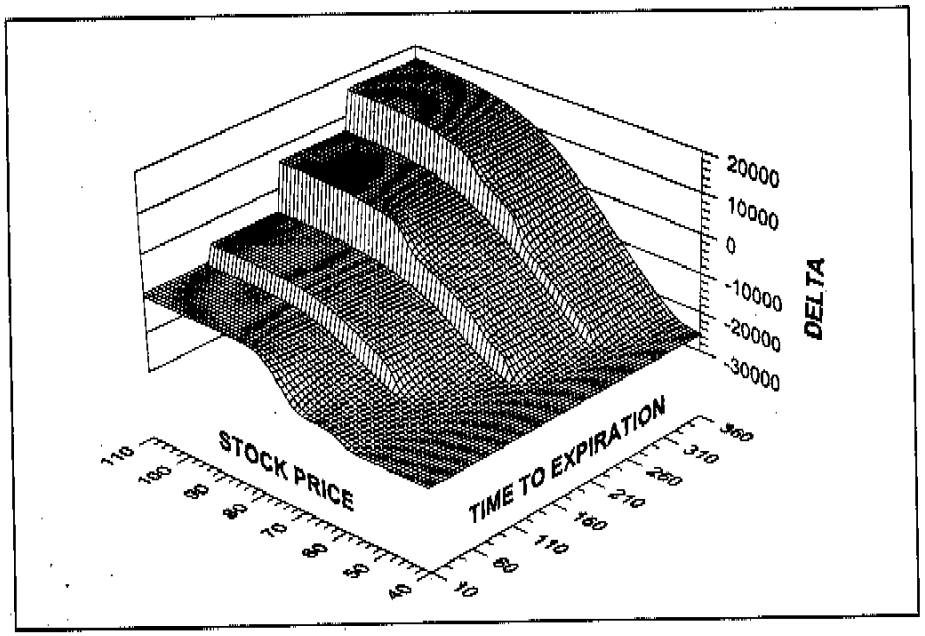


Figure 1c
Delta Risk



However, the portfolio's sensitivity to changes in volatility was not added for the options with different maturities. Simple adding of vega for options with different maturities is often incorrect as volatilities with different time horizons are not necessarily comparable. The relevant volatility for an option with 30 days to maturity is the future volatility for the next 30 days. For an option with 180 days to maturity, the future 180–day volatility is relevant.[1] Implied volatility, the market's estimate of future volatility, has historically been a reasonably good measure of future volatility (Beckers, 1981; Latané and Rendleman, 1976). A simple adding of vega for options with different maturities will be an accurate measure of volatility risk only if the following assumptions are met:

1. The correlation coefficients between volatilities across the term structure of volatility (volatilities for different maturities) are positive and close to one.

2. The term structure of volatility shifts in a parallel fashion, i.e., when the 30–day volatility increases by 4 percentage points, the 60–day volatility has to change by the same amount.

Table 2
Correlation Coefficients Between Historical Volatilities
(Computed from Daily Close Prices from 1/89–1/92 — 758 Observations)

Trading Days in Volatility Estimates	Coca–Cola	IBM	General Motors
20 and 40	0.80	0.73	0.77
20 and 60	0.69	0.66	0.72
20 and 120	0.44	0.43	0.58
60 and 80	0.92	0.90	0.93
60 and 120	0.76	0.72	0.85
60 and 250	–0.02	0.55	0.61
120 and 140	0.95	0.94	0.96
120 and 180	0.77	0.84	0.90
120 and 250	0.26	0.75	0.78
250 and 500	0.12	0.44	0.59

Table 2 shows an example of the correlation between volatilities for options on three large stocks, based on time periods of different length. The historical volatilities (sample standard deviation of the continuously compounded rate of return) are estimated on the basis of daily closing prices.[2] The volatility estimates were rolled over one day at a time. As expected, the correlation between most of the volatility estimates is relatively high. The correlation is greater the smaller the difference in length of the time periods used for the volatility estimates. 20–day to 40–day volatilities (Coca–Cola) have a correlation coefficient of 0.80, while 20 to 120–day have a correlation coefficient of only 0.44. Considering the fact that the different volatility estimates are based on the same underlying instrument, this *must* be the case. The first 20 close prices used in the calculation of the 40–day estimate are the same 20 close prices that make up the entire basis for the 20–day estimate. Thus, the shorter time estimate will constitute an increasing part of the longer time estimate the closer the volatility estimates are with regard to time (given that the longer estimate includes the shorter time period—as it should).

Weak correlations between historical volatilities can be explained by actual events which took place during the period after the expiry of the short–term volatility, but before the expiration of the long–term volatility.

Possible causes of week correlation between implied volatilities could be due to the expected occurrence of certain specific events after the expiration of some shorter options but during the life of longer options. Thus, some changes in

expected volatility may not affect short-term options at all but may have significant effect on longer options. The probability of singular events occurring after the short options maturity but during the life of longer options normally increases the longer the time there is between the options' expiration dates. The correlation between volatilities will be consequently less for options with considerable difference in date of expiration.

The second assumption, that the term structure of volatility shifts in a parallel fashion, has not proven tenable. Empirical studies (Burghardt and Lane, 1990) show that short-term volatility varies substantially more than long-term volatility. Figure 2 shows the volatility of historical volatilities for Coca-Cola, IBM, and General Motors, based on periods of different length. The figure clearly shows that the shorter the time period investigated, the greater the increase in volatility of volatility. The diagram is constructed on the basis of three years of historical closing prices (January 1, 1989–December 31, 1991; a total of 758 observations). By rolling over one day at a time, a total of 748 10-day historical volatility estimates are possible, 738 20-day estimates, and 508 one year estimates (250 trading days). During this period the one-year volatility of Coca-Cola showed a

Figure 2a
Coca-Cola
Volatility of Volatility Computed from Daily Close Prices from January 1989–January 1992

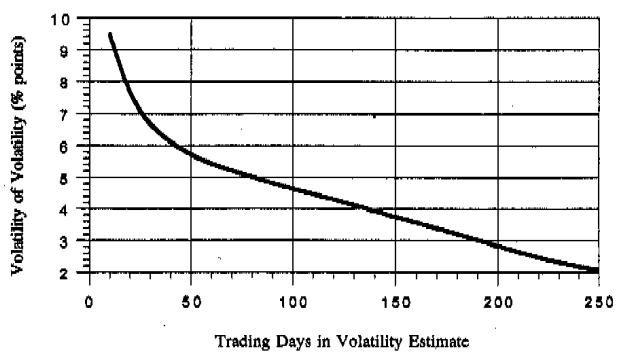


Figure 2b
International Business Machines

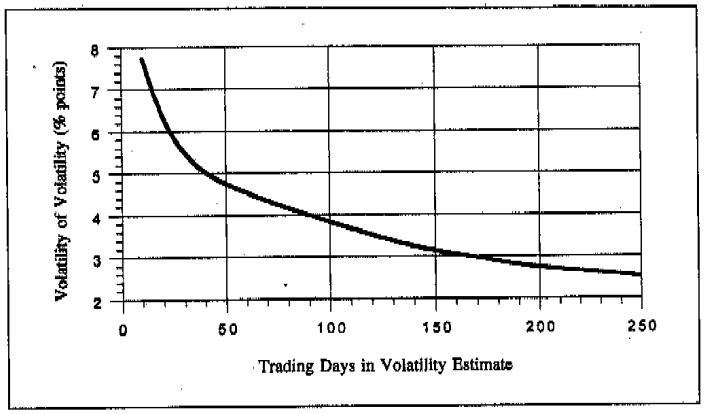
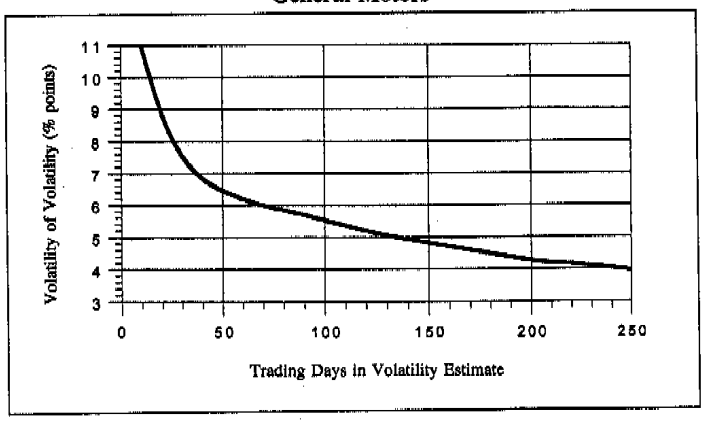


Figure 2c
General Motors



standard deviation slightly above 2 percentage points, while the 40–day volatility, for instance, showed a 6 percentage point standard deviation. In other words, the 40–day volatility was almost three times as volatile as the 250–day volatility.

The correlation coefficients in Table 2 and the volatility of volatilities in Figures 2a, b, and c, were calculated over an extended period of time (January 89–January 92). These values must accordingly be viewed as a type of average indicating what is "normal." Most market participants in the options markets, however, usually look at a far shorter time frame than three years. Traders in the options markets typically have a time frame varying from minutes, days, or months. It is therefore interesting to examine how correlation between different volatilities and volatilities of volatilities can alter over time. By calculating the correlation between 20 and 40 days historical volatility for Coca–Cola over a 20–day period and rolling that estimate through data material from the January 89–January 92 period, we find correlation coefficients as low as –0.803 and as high as 0.997. Correspondingly, the volatility of 40 days volatility calculated over a 40–day period in the 89–92 period varies from 0.58 to 7.25 percentage points.

From this we can conclude that the volatility curve does not have parallel shifts, and that the correlation between different volatilities is often less than perfect. Therefore, a simple summation of vega is not a good indicator of a portfolio's exposure to volatility. The following example illustrates this point.

Assume that we go long a call option with a strike price of 100 and time to maturity of 120 days. Simultaneously we go short a call option with an identical strike and 30 days to maturity. The stock price is 100. Both options have an implied volatility of 30 percent, and the risk–free interest rate is 10 percent. Using the Black and Scholes option pricing formula, this implies vega values of respectively 22.01 and –11.33. The option with the longest time to maturity has the highest absolute vega value—which indicates that it has the strongest sensitivity to changes in volatility. Adding vega for the two options results in a total vega exposure of +10.68. This means we are net long vega and will, according to theory, profit from an increase in volatility. Does this really happen?

If there is an increase of one percentage point for the option with 120 days to maturity, there will be a greater increase in the volatility of the option with 30 days to maturity. This is a result of short–term volatility being more volatile than long–term volatility. Let's assume that the 30–day volatility in this example is approximately one and a half times as volatile as the 120–day volatility. Typically, an increase from 30 percent to 31 percent in the 120–day volatility will result in an increase from 30 percent to 31.5 percent in the 30–day volatility. This presupposes a strong positive correlation between the volatilities. It would follow from this that the net result of the increase in volatility will be a profit of approximately +5.16 [$1 \times 22.01 + 1.5 \times (-11.23)$] rather than the 10.68 profit that we expected. Thus, the net vega exposure of the portfolio was misleading.

III. NET WEIGHTED VEGA EXPOSURE (NWV)

Although most market participants and academics have been aware of the fact that short–time volatility varies to a greater degree than long–term volatility, few have taken this into consideration when adding vega. For options with *the same time to maturity* and the same underlying instrument, but with different strike alternatives, a meaningful expression of the volatility exposure is provided by simple adding vega. As pointed out, this is not a viable method when the options have different maturities. By weighting the vega values of the different maturities with regard to the volatility of the volatility and adjusting for correlation, we can make different vega values commensurable. In this manner the weighted vega values can be added and applied in the same manner as for gamma, delta, and theta. The following formula may be used to find the net weighted vega exposure (NWV) of any option portfolio:

$$\mathrm{NWV} = \sum_{t=1}^{m} \sum_{i=1}^{n} w_{it}\, \mathrm{vega}_{it} \frac{\Psi_t}{\Psi_R} \rho_{\sigma(t),\sigma(R)}$$

where:

m = number of different maturities in the portfolio
n = number of different strikes with time to maturity t
vega_{it} = vega value of an option with strike i and time to maturity t
w_{it} = number of options with vega_{it}
Ψ_t = volatility of the volatility with time to maturity t
Ψ_R = volatility of the reference volatility
$\rho_{\sigma(t),\sigma(R)}$ = correlation between the volatility with time to maturity t and the reference–volatility

We can choose the volatility of one of the options included in the portfolio for the reference volatility. The time period of the reference volatility will change from day to day as the maturity of the relevant option approaches. The reference volatility might also be linked to a volatility estimate based on a period with constant length. The following example illustrates the use of NWV.

Let's assume that we own the portfolio shown in Table 3. How will the portfolio react to shifts in the term structure of volatility? To calculate NWV we need estimates of future volatility of volatilities with different maturities, and correlation coefficients between different volatilities. We chose a reference volatility equal to the volatility of the options with the longest time to maturity, i.e., 120–day volatility today, 119–day volatility tomorrow, etc. Let's say we have calculated the following historical volatilities of volatilities: 6.5 percentage points

Table 3
Option Portfolio

Days to maturity	120.00	60.00	60.00	30.00
Strike	$105.00	$85.00	$100.00	–$100.00
Call price	$4.99	$16.53	$4.88	$3.27
Vega	22.86	3.11	15.81	11.31
No. of contracts	450.00	100.00	–400.00	–300.00
Volatility of volatility	4.00%	5.50%	5.50%	6.50%
Correlation coefficients	1.00	0.85	0.85	0.65

Stock price $100, risk–free interest rate 10%, volatility 25%.
One contract represents rights to 100 shares of the underlying stock.

30–day, 5.5 percentage points 60–day, 4.0 percentage points 120–day, and correlation coefficients of 0.65 between 30–day volatility and the reference volatility (120–day), 0.85 for the 60–day volatility and, naturally, 1.0 between 120–day volatility and the reference volatility. If we assume that the historical estimates will not undergo significant changes in the coming period, these can be used without any adjustments. It follows that: $\Psi_1 = 6.5$, $\Psi_2 = 5.5$, $\Psi_3 = 4.0$, $\Psi_R = 4.0$, $\rho_{1R} = 0.65$, $\rho_{2R} = 0.85$, $\rho_{3R} = 1.0$.

$$\begin{aligned}
\text{NWV} &= \sum_{t=1}^{m} \sum_{i=1}^{n} w_{it}\ \text{vega}_{it}\ \frac{\Psi_t}{\Psi_R}\ \rho_{\sigma(t),\,\sigma(R)} \\
&= -300 \times 11.31 \times \frac{6.5}{4.0} \times 0.65 - 400 \times 15.81 \times \frac{5.5}{4.0} \times 0.85 \\
&\quad + 100 \times 3.11 \times \frac{5.5}{4.0} \times 0.85 + 450 \times 22.86 \times \frac{4.0}{4.0} \times 1.0 \\
&= -324.55
\end{aligned}$$

Hence, for each percentage point rise in the reference volatility (120–day), we will lose approximately $325. By simple adding vega, the net vega exposure would be:

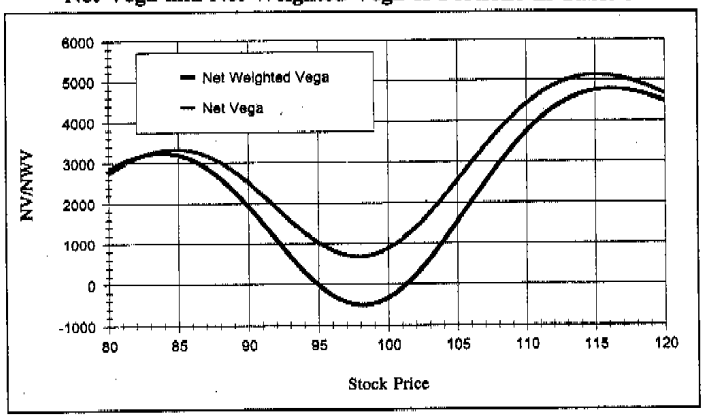
$$\text{Net Vega} = -300 \times 11.31 - 400 \times 15.81 + 100 \times 3.11 + 450 \times 22.86 = 881$$

The traditional vega measure tells us that for each percentage point rise in the volatility, we will earn approximately $881. Not weighting the vega value can thus lead to serious miscalculations of risk. Figure 3 shows a comparison of NWV and net vega (NV) for this option portfolio. NWV is lower than vega for most alternative stock prices, ceteris paribus. This is a direct result of short maturity options in the NWV measure (in this example) having larger weights than longer term options. Options in the traditional volatility measure all have equal weights.

It's the future volatility of the volatility and the future correlation coefficient, for the remaining time to maturity, which are relevant when determining the parameters included in the NWV formula. Since advance knowledge of these is impossible, a natural starting point is historical studies of the term structure of volatility. Adjustment of these estimates according to expectations for the future can be done, if deemed necessary.

The difference between vega and NWV will essentially depend upon the correlation between the volatilities and the volatility of the volatility. Subsection $\frac{\Psi_t}{\Psi_R}\rho_{\sigma(t),\sigma(R)}$ in NWV's formula plays a decisive role in this context. Since the correlation will usually be positive but less than perfect and the volatility of

Figure 3
Net Vega and Net Weighted Vega of Portfolio in Table 3



volatility empirically has been shown to be a decreasing function of time to maturity, these two factors will pull in opposite directions. Suppose, for example, that the future 30–day volatility of volatility for Coca–Cola is expected to be 7 percentage points and volatility of volatility of a 120–day option (reference option) is expected to be only 4 percentage points. The first part of subsection $\frac{\Psi_t}{\Psi_R}\rho_{\sigma(t),\sigma(R)}$ will then be $\frac{\Psi_t}{\Psi_R} = \frac{0.06}{0.04} = 1.5$. The higher volatility of volatility for the 30–day option results in a significant increase in the value of vega. A less than perfect correlation will pull in the opposite direction. If we suppose the correlation in the coming period will be $\rho_{\sigma(t),\sigma(R)} = 0.67$, the two subsections will counterbalance each other such that vega and NWV yield the same result. Use of NWV instead of simple vega will be for that reason most meaningful in instances where the correlation between the volatility estimates are expected to be relatively good and the volatility of volatility is expected to diverge considerably for the options with a shorter time to maturity than a lengthier time to maturity.

CONCLUSION AND SUMMARY

Different risk dimensions can be calculated using the chosen option pricing formula. Formal measures of the different risk dimensions have provided the participants in the market with a potent tool for risk analysis. With little difficulty, we have been able to add the different risk dimensions separately. Thus we can keep a general view of the risk even in large and complex portfolios of options.

Conversely, directly adding the derivative instruments' sensitivity to changes in volatility often gives a misleading picture of the real volatility risk. Net weighted vega solves this problem to a large extent. The formula for net weighted vega takes into account the fact that the term structure of volatility does not shift in a parallel fashion. Yet when using the NWV, it is important to be familiar with its assumptions and weaknesses.

A transition to net weighted vega will likely enable participants in the options markets to achieve even better estimates and analyses of volatility risk. To a large extent this will depend on how accurately we are able to estimate future correlations between volatilities with different times to maturity and future volatility of volatility. We hope that future research will answer these and other related questions concerning volatility risk.

APPENDIX

Black and Scholes option pricing formula:

$$C = SN(d_1) - Xe^{-rt}N(d_2)$$

$$P = Xe^{-rt}N(-d_2) - SN(-d_1)$$

$$d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)t}{\sigma\sqrt{t}}$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

Where the call option price (C) is a function of the underlying stock price (S), strike price (X), risk-free interest rate (r), time to expiration (t), and the annualized volatility (σ). $N(d_i)$ is the cumulative standard normal distribution function:

$$N(d_i) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{d_i} e^{-1/2z^2}\, dz$$

hence:

$$N'(d_i) = \frac{1}{\sqrt{2\pi}}\, e^{-1/2d_i^2}$$

is the standard normal density function.

Black and Scholes sensitivity to the different pricing parameters:

Delta:

$$\Delta_{Call} = \frac{\partial C}{\partial S} = N(d_1)$$

$$\Delta_{Put} = \frac{\partial P}{\partial S} = N(d_1) - 1$$

Gamma:

$$\Gamma_{\text{Call, Put}} = \frac{\partial^2 C}{\partial S^2} = \frac{\partial^2 P}{\partial S^2} = \frac{N'(d_1)}{S\sigma\sqrt{t}}$$

Theta:

$$\Theta_{\text{Call}} = -\frac{\partial C}{\partial t} = -\frac{SN'(d_1)\sigma}{2\sqrt{t}} - rXe^{-rt}N(d_2)$$

$$\Theta_{\text{Put}} = -\frac{\partial P}{\partial t} = -\frac{SN'(d_1)\sigma}{2\sqrt{t}} + rXe^{-rt}N(-d_2)$$

Vega:

$$\text{Vega}_{\text{Call, Put}} = \frac{\partial C}{\partial \sigma} = \frac{\partial P}{\partial \sigma} = S\sqrt{t}\,N'(d_1)$$

Rho:

$$\rho_{\text{Call}} = \frac{\partial C}{\partial r} = Xte^{-rt}N(d_2)$$

$$\rho_{\text{Put}} = \frac{\partial P}{\partial r} = -Xte^{-rt}N(-d_2)$$

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NOTES

1. To be precise: future volatility until the time when the option is expected to be exercised. American–type options can be expected to be exercised before maturity. It may be optimal to exercise a put option before maturity if it is sufficiently in–the–money. For call options where the underlying stock pays dividend, it may be optimal to exercise the option right before the stock goes ex–dividend.

2. For this study I have chosen daily closing prices for the calculation of historical volatilities. However, closing prices are merely arbitrarily chosen prices of one day's changes in stock prices. If the closing prices did not change for several days in a row, this would give zero volatility for the period in question—in spite of significant changes in stock prices during each day. Parkinson (1980), and later Garman and Klass (1980) and Beckers (1983) have shown that by using the high and the low, e.g., in combination with opening and closing prices, we get a more efficient measure of volatility. The aim of this article was to study the volatility risk of portfolios consisting of options. The exactitude of the volatility estimates then, is not of vital importance.

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