



THE COLLECTOR:

Why so Negative to Negative Probabilities?

What is the probability of the expected being neither expected nor unexpected?

1 The History of Negative Probability

In finance negative probabilities are considered nonsense, or at best an indication of model-breakdown. Doing some searches in the finance literature the comments I found on negative probabilities were all negative,¹ see for example Brennan and Schwartz (1978), Hull and White (1990), Derman, Kani, and Chriss (1996), Chriss (1997), Rubinstein (1998), Jorgenson and Tarabay (2002), Hull (2002). Why is the finance society so negative to negative probabilities? The most likely answer is simply that we “all” were taught that probabilities by definition must be between 0 and 1, as assumed in the Kolmogorov measure theoretical axioms. Our negativity to negative probabilities might be as short sighted as if we decided to limit ourselves to consider only positive money. I am not the first one to think that negative probabilities can be useful. Negative probabilities where to my knowledge first introduced by Paul Dirac. Dirac is probably best known for his mathematical prediction of antimatter, a feat for which he was awarded the Noble prize in 1933. In 1942 Paul Dirac wrote a

paper: “The Physical Interpretation of Quantum Mechanics” where he introduced the concept of negative energies and negative probabilities:

“Negative energies and probabilities should not be considered as nonsense. They are well-defined concepts mathematically, like a sum of negative money...”, Paul Dirac

The idea of negative probabilities has later got increased attention in physics and particular in quantum mechanics. Another famous physicist, Richard Feynman (1987) (also with a Noble prize in Physics), argued that no one objects to using negative numbers in calculations, although “minus three apples” is no valid concept in real life. Similarly he argued how negative probabilities as well as probabilities above unity can be very useful in probability calculations:

“Trying to think of negative probabilities gave me a cultural shock at first, but when I finally got easy with the concept I wrote myself a note so I wouldn’t forget my thoughts... It is usual to suppose that, since probabilities of events must be positive, a theory which gives negative numbers for such quantities must be absurd.”, Richard Feynman, 1987

Feynman discusses mainly the Bayes formula for conditional probabilities

$$P(i) = \sum_{\alpha} P(i|\alpha)P(\alpha),$$

where $\sum_{\alpha} P(\alpha) = 1$. The idea is that as long as $P(i)$ is positive then it is not a problem if some of the probabilities $P(i|\alpha)$ or $P(\alpha)$ are negative or larger than unity. This approach works well when one cannot measure all of the conditional probabilities $P(i|\alpha)$ or the unconditional probabilities $P(\alpha)$ in an experiment. That is, the variables α can relate to hidden states. Such an approach has therefore been used in quantum physics to solve problems involving hidden variables.

There has since been a multitude of papers in theoretical physics that focus on the use of negative probabilities. I have included a few examples in the reference list, that can also be downloaded from the web; Castro (2000), Cereceda (2000), Curtright and Zachosy (2001) and Peacock (2002). There seems to be a continuation of interest in negative probabilities in physics:

“I have done some work recently, on making supergravity renormalizable, by adding

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higher derivative terms to the action. This apparently introduces ghosts, states with negative probability. However, I have found this is an illusion. One can never prepare a system in a state of negative probability. But the presence of ghosts that one can not predict with arbitrary accuracy. If one can accept that, one can live quite happily with ghosts." Stephen Hawking

2 Negative Probabilities in Quantitative Finance

In this section we look at a few examples of where negative probabilities can show up in finance. My examples are not revolutionary in the sense that they solve problems never solved before. I hope, however, that they will illustrate why negative probabilities are not necessary "bad", and that they might even be useful.

2.1 Negative Probabilities in the CRR binomial tree

The well-known Cox, Ross, and Rubinstein (1979) binomial tree (CRR tree) is often used to price a variety of derivatives instruments, including European and American options. The CRR binomial tree can be seen as a discretization of geometric Brownian motion $dS = \mu S dt + \sigma S dz$, where S is the asset price, μ is the drift and σ is the volatility of the asset. In the CRR tree the asset price in any node of the tree is given by

$$Su^i d^{j-i}, \quad i = 0, 1, \dots, j,$$

where the up and down jump size that the asset price can take at each time step Δt apart is given by

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}},$$

where $\Delta t = T/n$ is the size of each time step, and n is the number of time steps. The layout of the asset price (the geometry of the tree) we will call the sample space (the set of all asset price S_{ij} node values). The probability measure P is the set of the probabilities at the various nodes. The probabilities related to each node in the CRR tree follows from the arbitrage principle

$$Se^{b_i \Delta t} = p_i u S + (1 - p_i) d S, \quad (1)$$

where p_i is risk-neutral probability of the asset price increasing at the next time step, and b_i is the cost-of-carry of the underlying asset.² The i

subscript indicates that we can have a different probability and also cost-of-carry at each time step. Solving 1 for p_i we get

$$p_i = \frac{e^{b_i \Delta t} - d}{u - d}.$$

The probability of going down must be $1 - p_i$ since the probability of going either up or down equals unity. As mentioned by Chriss (1997) a low volatility and relatively high cost of carry in the CRR tree can lead to negative risk-neutral probabilities.³ More precisely we will have negative probabilities in the CRR tree when⁴

$$\sigma < |b_i \sqrt{\Delta t}|.$$

In this context it is worth mentioning that even if the CRR binomial tree can give negative and higher than unity probabilities the sum of the down and up probability will still sum to one.

2.1.1 What to do with negative probabilities?

Assume we are using a CRR binomial tree to value a derivative instrument, and consider the following numerical example. The asset price is 100, the time to maturity of the derivative instrument is 6 months, the volatility of the underlying asset is 2%, the cost-of-carry of the underlying asset is 12% for the first month, and are increasing by 0.5% for every month thereafter. For simplicity we use only six time steps. That is $S = 100$, $T = 0.5$, $b_1 = 0.12$, $\sigma = 0.02$, $n = 6$. From this we have $\Delta t = 0.5/6 = 0.833$ and

$$u = e^{0.02\sqrt{0.833}} = 1.006,$$

$$d = e^{-0.02\sqrt{0.833}} = 0.9942.$$

Further we get a risk-neutral up probability of

$$p_1 = \frac{e^{0.12 \times 0.833} - 0.9942}{1.006 - 0.9942} = 1.3689.$$

and we get a down probability of

$$1 - p_1 = 1 - 1.3689 = -0.3689$$

As expected, we get negative probabilities. In this generalized version of the CRR tree we can have different probabilities for every time steps, so we could have probabilities outside the interval $[0, 1]$ for some time steps and inside for other time steps, in numerical example all the probabilities

will be outside zero and one. When we observe negative probabilities in a tree model we have at least three choices

1. We can consider negative probabilities as unacceptable, and any model yielding negative probabilities as having broken down. The model should be trashed, or alternatively we should only use the model for input parameters that do not result in negative probabilities.
2. Override the negative probabilities. Basically replacing any transition probabilities that are negative or above unity with probabilities consistent with the standard axiomatic framework. For example Derman, Kani, and Chriss (1996) suggest this as a possible solution in their implied trinomial tree when running into negative probabilities.
3. Look at negative probabilities as a mathematical tool to add more flexibility to the model.

From these choices I have only seen choice 1 (most common) and 2 being discussed in the finance literature. I am not saying that choice 1 and 2 are wrong, but this should not automatically make us reject choice 3.

Back to the CRR binomial tree example. In this case what we do about the negative probabilities should depend on the use of the model. Let's consider case 1. In this case we would simply say that for this numerical case the model is useless. In the CRR model case we can actually get around the whole problem of negative probabilities, but let us ignore this for the moment.

In case 2 we could override the few negative probabilities in some "smart" way. This will in general make the model loose information (in particular if calibrated to the market in some way) and is not desirable, as also indicated by Derman, Kani, and Chriss (1996) in use of their implied trinomial tree.

Consider now case 3. The problems with the CRR tree is in my view actually not the negative probabilities, but the choice of sample space. The negative probabilities simply indicate that the forward price is outside the sample space. The forward price is in a "hidden state", not covered by the sub-optimal location of nodes in the tree. The sub-optimal choice of sample space could certainly be a problem when trying to value some options.

Actual realizations of the asset price of relevance to the option price could easily fall outside the sample-space of the model. The strike price X can also be outside the sample space of the tree. This seems to be the main problem with the CRR tree, and not the negative probabilities. We could easily think of an example where the user of the CRR model simply used it to compute forward prices, taking into account a deterministic term structure of interest rates and dividends (b). Computing the forward price can naturally be achieved in a simpler and more efficient way, but there is nothing wrong with using a complex model to value a simple product, except possibly wasting computer speed. Even with negative probabilities (and probabilities larger than unity) the CRR tree will still give the correct forward price. Allowing negative probabilities still leaves the model partly intact. The negative probabilities actually seems to add flexibility to the model.

It is also worth refreshing our memories that most probabilities used in quantitative finance are so called risk-neutral probabilities, including the binomial probabilities we just have looked at. As every reader probably already know, risk-neutral probabilities⁵ should not be confused with real probabilities. Risk neutral probabilities sometimes also called pseudo-probabilities are simply computational devices constructed for a "fantasy world" (in this case a risk-neutral world) to simplify our calculations. For this reason I can see nothing wrong with at least starting to thinking about negative probabilities as a mathematical tool of convenience. This expand to outside the world of binomial trees, a binomial model can simply be seen as a special case of a explicit finite difference model, see for example Heston and Zhou (2000) or James (2003). In the same way a trinomial tree can also be shown to be equivalent to the explicit finite difference method (see for example James (2003)) when the probability of the asset price going up p_u and down p_d and stay at the same are set to⁶

$$p_u = \frac{1}{6} + (b - \sigma^2/2)\sqrt{\frac{\Delta t}{12\sigma^2}}$$

$$p_d = \frac{1}{6} - (b - \sigma^2/2)\sqrt{\frac{\Delta t}{12\sigma^2}}$$

$$p_m = \frac{2}{3}$$

This will lead to negative up probability p_u if

$$\sigma > \sqrt{2b + \frac{2}{3\Delta t} + \frac{2\sqrt{1+6b\Delta t}}{3\Delta t}}$$

and negative down probability p_d if

$$\sigma < \sqrt{2b + \frac{2}{3\Delta t} - \frac{2\sqrt{1+6b\Delta t}}{3\Delta t}}$$

For example with cost-of-carry 20% and 20 time steps and one year to maturity, then we will get a negative down probability if the volatility is below 7.63%, this is far from a totally unrealistic case. Similarly we can also get probabilities higher than unity⁷. In the same way a CRR equivalent trinomial tree can also give negative probabilities, see appendix.

If you take a close look at the Collector cartoon story accompanying this article you will see it is surprisingly similar to the binomial and trinomial tree just described. By believing he is talking to James, the professor has no idea that the expected death of James is far outside his sample space. The professor is selecting a sub-optimal sample space, only the future, as basis for his model. The Collector, admittedly having more information is recognizing that the professor uses a sub-optimal sample space, but is still able to answer the question with remarkable accuracy, by allowing negative probabilities (pseudo probabilities). This is very much a parallel to considering a node of a tree model outside the geometry of the tree.

3 Getting the Negative Probabilities to Really Work in Your Favor

In the case of the discretization of a geometric Brownian motion through a binomial or trinomial model we can admittedly avoided negative probabilities all together. In the CRR or trinomial tree just described this can be done simply by respectively setting the number of time steps n equal to or higher than

$$\text{Integer} \left[\frac{T}{(\sigma/b)^2} \right],$$

$$\text{Integer} \left[3bT \left(-1 + \frac{b}{\sigma^2} + \frac{\sigma^2}{4b} \right) \right] + 1.$$

For very low volatility and high cost-of-carry this will typically require a lot of time steps, and can for this reason be very computer intensive. A better way to avoid negative probabilities is to chose a more optimal sample space, for example in a binomial tree this can be done using a geometry of the tree as suggested by Jarrow and Rudd (1983), setting

$$u = e^{(b-\sigma^2/2)\Delta t + \sigma\sqrt{\Delta t}},$$

and

$$d = e^{(b-\sigma^2/2)\Delta t - \sigma\sqrt{\Delta t}},$$

which gives equal up and down probability of

$$p = 0.5, \quad 1 - p = 0.5.$$

Even if the CRR tree and the Jarrow-Rudd tree use different sample space and probability measure they are both equivalent in the limit, for many time steps. For a binomial tree there is a almost an unlimited amount of sample spaces to choose from, each with their own probability measure, but all leading to the same result in the limit.

For more complex models and stochastic processes we will not necessary be able to avoid negative probabilities. Does this mean that the model has to be trashed? In the introduction to this article I basically told you that all papers I have seen in the finance literature are negative to negative probabilities. This is not entirely true; in a recent and very interesting paper by Forsyth, Vetzal, and Zvan (2001): "Negative Coefficients in Two Factor Option Models" the authors illustrate how some finite difference/finite element models with negative probabilities are still stable and consistent. They moreover show how the requirement of positive pseudo-probabilities not only are unnecessary but can even be detrimental. Forsyth, Vetzal and Zvan seem to try to avoid the term "negative probability", and are consistently using the term "negative coefficients". What they call negative coefficients are closely related to the binomial and trinomial probabilities we just discussed (as also indicated by the authors). That the authors possibly try to avoid the term negative probability seems to be somewhat a parallel to how many physicist treat the Wigner distribution.

Wigner and Szilard developed a distribution function which for the first time was applied by Wigner to calculate quantum corrections to a gas pressure formula. Wigner original called this a probability function. The Wigner distribution gives apparently negative probabilities for some quantum states. For this reason many physicists prefer to call it the Wigner function, this even if it has no other physical interpretation than a probability distribution. Khrennikov (1999) is giving a detailed mathematical description of why it should be considered a probability distribution, and also explain why the Wigner distribution results in negative probabilities.

Even if Forsyth, Vetzal and Zvan seems to avoid the term negative probabilities, their paper are in an excellent way illustrating just what I have been looking for, a demonstration that allowing negative probabilities really can make a difference in quantitative finance models.

The binomial, trinomial and finite difference models we have touched upon so far were originally developed by assuming probabilities in the interval $[0, 1]$. So even if negative probabilities seems to add flexibility to some of these models we must be careful with the interpretation of any such negative pseudo-probabilities.

4 Hidden Variables in Finance

Feynman's idea of applying negative probabilities to hidden variables can possibly have some parallels in finance. Expected return, and expected risk (expected volatility, correlation etc.) are hidden variables that can not be observed directly. Such hidden variables in finance still play an important role and can affect other observable variables. Derman (1996) describe these as hidden variables in the context to the somewhat related topic of model risk, where he indicate that such variables simply may be individuals' opinion. Much of modern financial modeling deals with uncertainty for such hidden variables, still it looks like nobody have considered including negative probabilities in such calculations. Inspired by Dirac, Feynman, and Hawking could there be that negative probabilities also could be of use to model hidden financial ghosts? Ultimately one would naturally like to see research that proves that using negative probabilities gives a clear advantage.

4.1 Negative Probabilities Hidden in Waste from Model Break Downs?

Personally I don't know of any financial model that at some point in time has not had a breakdown—there's a reason we call them "models". Just as an example, any derivatives valuation model I am aware of assume that the current asset price is known. When valuing for example a stock option we take the latest traded stock price from our computer screen as input to the model. In reality this price is not known. Every year there are multiple incorrectly reported prices, due to a human error, or a hardware or software problem. One can naturally reduce the uncertainty surrounding the current price by double-checking several independent providers etc. Still there are uncertainty concerning any price, even the current one. In some sense almost any asset in financial markets can be affected by "hidden variables". For example in the interbank Foreign Exchange markets, if you physically trade an option at a price over the phone you can not with 100% certainty know that you actually traded at that price. It is not uncommon that the counterpart calls you up hours later to cancel or adjust the price of a deal, because they claim they priced it using wrong input parameters, human error or whatever. Few or no models in quantitative finance take such situations into account.

Others examples of model break downs we have when asset prices takes values completely outside our expected sample space. I remember some years back in time, when the electricity price in a local area of Norway actually went negative for a few hours during the night. The electricity was produced by hydropower using large turbines. When there is an over supply of electricity on the grid one can naturally shut down some of the turbines, however this can be a costly process. In this case, instead of shutting down the turbine the power producer was willing to pay someone to use the electricity. The Scandinavian electricity markets are probably the most well functioning electricity markets in the world, including futures, forwards and options. We can easily think about a derivative model assuming positive prices breaking down in

such a scenario. Finance Professors' assumptions of "no free lunch" is probably only a reality at the university campuses.

My main point is that there are almost unlimited examples of cases where the real uncertainty is outside the sample space of our quantitative models. A whole new field of finance has grown out from model errors, going under the name "model risk". Can there be that negative probabilities are hidden in "data waste" from such model breakdowns? There is obvious many ways we can improve our models and still stay inside the standard assumption of probabilities between zero and one, but could it be that building our models from the ground up to allow negative probabilities in the first place could help us make the models more robust, closer to reality?

5 The Future of Negative Probabilities in Quantitative Finance

The main reason that negative probabilities have not found much use in quantitative finance yet is probably that the researchers that have developed these models have been doing this under the belief that any probability must lie between zero and one, limiting their view to Kolmogorov probabilities. So far, I have just done a feeble attempt to trace the footsteps of Dirac and Feynman. That is we have considered negative probabilities as just formal quantities that can be useful in certain calculations. Most people in quantitative finance are interested in finance and not the foundation of probability theory and have probably for this reason ignored that the Komologrov model is not necessary the complete picture of stochastic reality. To take the next step we need to use a rigid mathematical foundation for a probability theory that also allows for negative probabilities. Andrei Khrennikov has developed such a theory, he has found the root of negative probabilities in the very foundation of probability theory: ensemble and frequency. With background in his theory described in mathematical detail in his brilliant book: "Interpretations of Probability" Khrennikov (1999), see also Khrennikov (1997), it is reason to believe that negative probabilities applied to finance could take it's next step? The

Kolmogorov probability theory that is the basis of all mathematical finance today can be seen as a limited case of a more general probability theory developed by a handful of scientists standing on the shoulder of giants like Dirac and Feynman. The probability that the future of quantitative finance will contain negative probabilities is possibly negative, but that does not necessary exclude negative probabilities.

If you eliminate the impossible, whatever remains, however improbable, must be the truth. Sherlock Holmes

6 Appendix: Negative probabilities in CRR equivalent trinomial tree

Trinomial trees introduced by Boyle (1986), are similar to binomial trees and are also very popular in valuation of derivative securities. One possibility is to build a trinomial tree with CRR equivalent parameters, where the asset price at each node can go up, stay at the same level, or go down. In that case, the up-and-down jump sizes are:

$$u = e^{\sigma\sqrt{2\Delta t}}, \quad d = e^{-\sigma\sqrt{2\Delta t}},$$

and the probability of going up and down respectively are:

$$p_u = \left(\frac{e^{b\Delta t/2} - e^{-\sigma\sqrt{\Delta t/2}}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}} \right)^2$$

$$p_d = \left(\frac{e^{\sigma\sqrt{\Delta t/2}} - e^{b\Delta t/2}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}} \right)^2.$$

The probabilities must sum to unity. Thus the probability of staying at the same asset price level is

$$p_m = 1 - p_u - p_d.$$

When the volatility σ is very low and the cost-of-carry is very high p_u and p_d can become larger than one and then naturally p_m will become negative. More precisely we will get a negative probability $p_m < 0$ when

$$\sigma < \sqrt{\frac{b^2 \Delta t}{2}}.$$

From this we can also find that we need to set the number of time steps to $n \geq \text{Integer} \left[\frac{b^2 T}{2\sigma^2} \right] + 1$ to avoid negative probabilities. Alternatively we could avoid negative probabilities by choosing a more optimal sample space.



FOOTNOTES & REFERENCES

- At least in the sense that they urge one to avoid negative probabilities.
- For stocks ($b = r$), stocks and stock indexes paying a continuous dividend yield q ($b = r - q$), futures ($b = 0$), and currency options with foreign interest rate r_f ($b = r - r_f$).
- Negative risk-neutral probabilities will also lead to negative local variance in the tree, the variance at any time step in the CRR tree is given by $\sigma_t^2 = \frac{1}{\Delta t} p(1-p)(\ln(u))^2$. Negative variance seems absurd, there is several papers discussing negative volatility, see Peskir and Shiryaev (2001), Haug (2002) and Aase (2004).
- This is also described by Hull (2002) page 407, as well as in the accompanying solution manual page 118.
- Risk-neutral probabilities are simply real world probabilities that have been adjusted for risk. It is therefore not necessary to adjust for risk also in the discount factor for cash-flows. This makes it valid to compute market prices as simple expectations of cash flows, with the risk adjusted probabilities, discounted at the riskless interest rate—hence the common name “risk-neutral” probabilities, which is somewhat of a misnomer.
- Hull (2002) also shows the relationship between trinomial trees and the explicit finite difference method, but using another set of probability measure, however as he points out also with his choice of probability measure one can get negative probabilities for certain input parameters.
- We will get a up probability p_u , higher than unity if

$$\sigma < \sqrt{2b + \frac{50}{3\Delta t} - \frac{10\sqrt{25 + 6b\Delta t}}{3\Delta t}},$$

and down probability of higher than unity if

$$\sigma > \sqrt{2b + \frac{50}{3\Delta t} + \frac{10\sqrt{25 + 6b\Delta t}}{3\Delta t}}.$$

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